# Appendix 2 : Determinants and Simultaneous Equations

## (a) 2×2 Determinants

Value:

The value of a 2×2 determinant is given by : $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)$

or, using a slightly different notation $\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = (m_{11}m_{22} - m_{12}m_{21})$

Example:

What is the value of the determinant : $\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$ ?

$$\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = (2\times 3) - (1\times 1) = 6 - 1 = 5$$

Properties:

Multiplying each and every term of the determinant by a common factor, is the same as multiplying the determinant by the same factor taken to a power equal to the order of the determinant (in this case 2).

i.e. $$\begin{vmatrix} ya & yb \\ yc & yd \end{vmatrix} = y^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

*Proof* :

$$\begin{vmatrix} ya & yb \\ yc & yd \end{vmatrix} = (ya\times yd) - (yb\times yc) = y^2[(a\times d) - (b\times c)] = y^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

## (b) 3×3 Determinants

Value:

The value of a 3×3 determinant is given by :

$$\begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} = i\begin{vmatrix} b & c \\ e & f \end{vmatrix} - j\begin{vmatrix} a & c \\ d & f \end{vmatrix} + k\begin{vmatrix} a & b \\ d & e \end{vmatrix} = i(bf - ce) - j(af - cd) + k(ae - bd)$$

or, using a slightly different notation

$$\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} = m_{11}\begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} - m_{12}\begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} + m_{13}\begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$$

$$= m_{11}(m_{22}m_{33} - m_{23}m_{32}) - m_{12}(m_{21}m_{33} - m_{23}m_{31}) + m_{13}(m_{21}m_{32} - m_{22}m_{31})$$

Example:

What is the value of the determinant : $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix}$ ?

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 2(2-0) - 1(2-0) + 1(0-1) = 4 - 2 - 1 = 1$$

### (c) *n*×*n* Determinants

Value:

The value of a *n*×*n* determinant is given by progressively breaking it down into smaller determinants using the procedure outlined below:

$$\begin{vmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{21} & m_{22} & m_{23} & & m_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ m_{n1} & m_{n2} & m_{n3} & \cdots & m_{nn} \end{vmatrix} = m_{11}\begin{vmatrix} m_{22} & m_{23} & \cdots & m_{2n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ m_{n2} & m_{n3} & \cdots & m_{nn} \end{vmatrix} - m_{12}\begin{vmatrix} m_{21} & m_{23} & \cdots & m_{2n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ m_{n1} & m_{n3} & \cdots & m_{nn} \end{vmatrix} + \ldots\ldots.etc.$$

where the sign preceding each term alternates between + and – , and the terms themselves are formed by taking each element from the top row and multiplying it by the smaller determinant formed by excluding the row and column in which this element occurs.

Properties:

Multiplying each and every term of the determinant by a common factor, is the same as multiplying the determinant by the same factor taken to a power equal to the order of the determinant (in this case *n*).

i.e.
$$\begin{vmatrix} ym_{11} & ym_{12} & ym_{13} & \cdots & ym_{1n} \\ ym_{21} & ym_{22} & ym_{23} & & ym_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ ym_{n1} & ym_{n2} & ym_{n3} & \cdots & ym_{nn} \end{vmatrix} = y^n \begin{vmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{21} & m_{22} & m_{23} & & m_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ m_{n1} & m_{n2} & m_{n3} & \cdots & m_{nn} \end{vmatrix}$$

### (d) Solving Simultaneous Equations

Consider a set of simultaneous equations of the form:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n = b_3$$

*etc.*

These equations may also be represented in a matrix form as indicated below:

$$\mathbf{AX} = \mathbf{B}$$

where :

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} , \quad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \end{pmatrix} , \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \end{pmatrix}$$

The desired solutions are the set of $x_i$ values that satisfy these equations.

The approach to obtaining the solutions when using the matrix representation is to pre-multiply both sides of the matrix equation by $\mathbf{A}^{-1}$, the inverse of matrix $\mathbf{A}$. This gives,

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$$

Now , $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix (the matrix equivalent of multiplying by one)

Hence it follows (since $\mathbf{IX} = \mathbf{X}$) that the matrix $\mathbf{X}$ (corresponding to the required set of $x_i$ values) may simply be obtained from:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

*What is the inverse matrix* $\mathbf{A}^{-1}$ ?

It may be shown that

$$\mathbf{A}^{-1} = \frac{adj(\mathbf{A})}{|\mathbf{A}|}$$

where:

$adj(\mathbf{A})$ is the adjoint matrix for the matrix $\mathbf{A}$.

$|\mathbf{A}|$ (or $det(\mathbf{A})$ ) is the determinant of the matrix $\mathbf{A}$.

*What happens if* $\mathbf{B} = 0$ (i.e. <u>all</u> the $b_i$ values are equal to zero) ?

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{adj(\mathbf{A})}{|\mathbf{A}|} \cdot \mathbf{0}$$

There are two possible solutions:

(i) $\mathbf{X} = \mathbf{0}$ i.e. <u>all</u> the $x_i$ values are equal to zero.

or

(ii) $|\mathbf{A}| = 0$ i.e. the determinant is zero.

The first is the trivial solution (not of much interest), whilst the second defines the condition which must be satisfied for a set of non-trivial $x_i$ values to exist (but does not actually provide these solutions).

*Example :*

Consider the simultaneous equations:

$$\begin{aligned} (2-E)x_1 + 2x_2 &= 0 \\ x_1 + (1+E)x_2 &= 0 \end{aligned}$$

which in matrix form becomes

$$\begin{pmatrix} (2-E) & 2 \\ 1 & (1+E) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

The trivial solutions are $x_1 = 0$ and $x_2 = 0$ .

Non-trivial solutions only exist when

$$\begin{vmatrix} (2-E) & 2 \\ 1 & (1+E) \end{vmatrix} = 0$$

$$(2-E)(1+E) - 2 = 0$$

$$\Rightarrow \quad -E^2 + E = 0$$

$$\Rightarrow \quad E\,(1-E) = 0$$

$$\Rightarrow \quad E = 0 \quad \text{or} \quad (1-E) = 0$$

$$\Rightarrow \quad E = 0 \quad \text{or} \quad E = 1$$

What are the non-trivial solutions ?

(i) If $E = 0$ , the simultaneous equations become

$$\begin{aligned} 2x_1 + 2x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$$

i.e. any pairs of values such that $x_1 = -x_2$

(ii) If $E = 1$ , the simultaneous equations become

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_1 + 2x_2 &= 0 \end{aligned}$$

i.e. any pairs of values such that $x_1 = -2x_2$

Note that,

1. the non-trivial solutions correspond to those situations where the simultaneous equations become equivalent.
2. unique solutions (i.e. specific values of $x_1$ and $x_2$ ) only arise if additional constraints are applied, such as the normalisation condition in quantum mechanics.